MATH 220.204, APR 1 2019

1. Let ζ denote the Riemann zeta function, and suppose that $\zeta(z) = 0$ for some $z \in \mathbb{C}$. Prove that $\mathrm{Re}(z) = 1/2$.

   This is known as the Riemann Hypothesis. It is still a major unsolved question connecting complex analysis to number theory.

2. Prove that every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

   This was known as the Poincare Conjecture but is now a theorem, primarily due to Perelman.

3. Let X be a nonsingular complex projective manifold. Then every Hodge class $\alpha \in \mathrm{Hdg}^k(X) = H^{2k}(X;\mathbb{Q}) \cap H^{k,k}(X)$ is a rational linear combination of algebraic cocycles.

   This is the Hodge Conjecture, a major unsolved question in algebraic geometry.

4. Prove that for any compact simple gauge group G, a non-trivial quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has mass gap $\Delta > 0$.

   This is the Yang-Mills existence and mass gap problem, a major unsolved question in mathematical physics.

5. Let $\mathcal{E}$ be an elliptic curve over a number field K, and let $L(E, s)$ be the associated L-function. Prove that the rank of $E(K)$ is the order of the zero of $L(E, s)$ at $s = 1$.

   This is a small part of the Birch and Swinnerton–Dyer conjecture, an unsolved question in number theory.

APRIL FOOL'S!

1. Consider the sequence defined by:

$$\begin{cases} u_0 = \dfrac{1}{2} \\ u_{n+1} = \dfrac{u_n + 1}{u_n + 2} & \text{for } n \geq 0. \end{cases}$$

Prove that $0 < u_n < \frac{2}{3}$ for every integer $n \geq 0$.

We prove the result by induction on n.

Base Case: $n = 0$. This is immediate because $0 < \frac{1}{2} < \frac{2}{3}$.

Inductive Step: Suppose that $0 < u_n < \frac{2}{3}$ for some n. Since $u_{n+1} = \frac{u_n+1}{u_n+2}$, we wish to show that $0 < \frac{u_n+1}{u_n+2} < \frac{2}{3}$.

- Because $u_n > 0$, it follows that $u_n + 1 > 0$ and $u_n + 2 > 0$, it follows that $\frac{u_n+1}{u_n+2} > 0$, which proves the first inequality.
- Because $u_n < \frac{2}{3}$, it follows that $u_n < 1$ and thus

$$\begin{aligned} u_n &< 1 \\ 3u_n + 3 &< 2u_n + 4 \\ 3(u_n + 1) &< 2(u_n + 2) \\ \frac{u_n + 1}{u_n + 2} &< \frac{2}{3} \end{aligned}$$

as desired.

Thus, $0 < \frac{u_n+1}{u_n+2} < \frac{2}{3}$, which completes the induction.

2. Let n be a positive integer, and let $\mathbb{Z}_n$ be the set of integers modulo n. Let

$$S = \{[x] \in \mathbb{Z}_n : [x^2] = [x]\}$$

(a) Write out the elements of S when $n = 15$. You may write this as a list $\{[a], [b], [c], \ldots\}$.

When $n = 15$, we have $S = \{[0], [1], [6], [10]\}$.

(b) Prove that if n is prime, then $S = \{[0], [1]\}$.

For any integer x, we have

$$\begin{aligned} = [x] &\iff x^2 \equiv x \pmod{n} \\ &\iff n | x^2 - x \\ &\iff n | x(x-1) \end{aligned}$$

Since n is prime, $n|x(x-1)$ if and only if $(n|x \vee n|(x-1))$, which is true if and only if $([x] = [0] \vee [x] = [1])$.

3. Let F_n be the Fibonacci sequence:

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{if } n > 2.$$

Prove that $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$.

We use induction on n. The base case is $n = 1$, which is immediate because $F_1^2 = 1 = F_1 F_2$.

Suppose that $\sum\limits_{k=1}^{n} F_k^2 = F_n F_{n+1}$. We wish to show that $\sum\limits_{k=1}^{n+1} F_k^2 = F_{n+1} F_{n+2}$. We have that

$$\begin{aligned}\sum_{k=1}^{n+1} F_k^2 &= F_{n+1}^2 + \sum_{k=1}^{n} F_k^2 \\ &= F_{n+1}^2 + F_n F_{n+1} \\ &= F_{n+1}(F_{n+1} + F_n) \\ &= F_{n+1} F_{n+2}\end{aligned}$$

which completes the induction.

4. Suppose that $f : A \to B$ is a function and $C \subseteq B$. Prove that $f(f^{-1}(C)) = C \cap f(A)$.

First we show that $f(f^{-1}(C)) \subseteq C \cap f(A)$. Suppose that $y \in f(f^{-1}(C))$. Then there is some $x \in f^{-1}(C)$ such that $f(x) = y$. Since $f^{-1}(C) \subseteq A$, it immediately follows that $f(x) \in f(A)$. Since $x \in f^{-1}(C)$, it immediately follows that $f(x) \in C$. Thus, $f(x) \in C \cap f(A)$. So $y \in C \cap f(A)$.

Suppose that $y \in C \cap f(A)$. Since $y \in f(A)$, it follows that the set $f^{-1}(\{y\})$ is nonempty, so pick any $x \in f^{-1}(\{y\})$. Since $y \in C$, it immediately follows that $x \in f^{-1}(C)$. Thus, $y \in f(f^{-1}(C))$.

5. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by

$$\forall n \in \mathbb{N}, \quad f(2n-1) = 3n - 2 \qquad f(2n) = 3n - 1$$

Prove that $\mathbb{N} \times \mathbb{N}$ is denumerable by showing that the function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined as $g(m, n) = 3^{m-1} f(n)$ is bijective.

g is injective: Suppose that $a, b, c, d \in \mathbb{N}$ are such that $3^{a-1} f(b) = 3^{c-1} f(d)$. Then, $f(b) = 3^{c-a} f(d)$. Neither $f(b)$ nor $f(d)$ can be a multiple of 3, so it follows that $\boxed{a = c}$. Thus, $f(b) = f(d)$. The function f is increasing, because for every n, $f(2n-1) = 3n-2$ is less than $f(2n) = 3n-1$ is less than $f(2n+1) = 3n+1$. Therefore, f is injective and so $\boxed{b = d}$.

g is surjective: We will prove by strong induction on the natural number N, that N is in the image of the function g. The base case $N = 1$ is true because $g(1, 1) = 1$.

We now prove the inductive step. Let $N > 1$, and suppose that every positive integer less than N is in the image of the function g.

- If $N \equiv 1 \pmod 3$, then there is some $n \in \mathbb{N}$ such that $N = 3n - 2$, in which case $g(0, n) = N$.
- If $N \equiv 2 \pmod 3$, then there is some $n \in \mathbb{N}$ such that $N = 3n - 1$, in which case $g(0, n) = N$.
- If $N \equiv 0 \pmod 3$, then $N/3$ is a positive integer and $N/3 < N$. By the inductive hypothesis, there are positive integer m, n such that $g(m, n) = N/3$. Then $g(m + 1, n) = N$.